

A LIOUVILLE PROPERTY FOR GRADIENT GRAPHS AND A BERNSTEIN PROBLEM FOR HAMILTONIAN STATIONARY EQUATIONS.

ABSTRACT. Using an rotation of Yuan, we observe that the gradient graph of any semiconvex function is a Liouville manifold, that is, does not admit bounded harmonic functions. As a corollary, we find that any solution of the fourth order Hamiltonian stationary equation satisfying

$$\theta \geq (n-2) \frac{\pi}{2} + \delta$$

for some $\delta > 0$ must be a quadratic.

In this short note, we record the following.

Theorem 1. *Suppose that u is semi-convex. Then the gradient graph*

$$\Gamma = \{(x, Du(x)) : x \in \mathbb{R}^n\}$$

with the induced submanifold metric, is a Liouville manifold.

Recall that a manifold has the Liouville property if all bounded harmonic functions are constant. A rotation of Yuan [Yua02] allows us to write the Laplace operator as a uniformly elliptic divergence operator. The result then follows readily from the De-Giorgi-Nash-Moser theory.

We are interested in studying a fourth order special Lagrangian type equation. Let λ_i be the eigenvalues of the Hessian D^2u . The lagrangian phase is given by

$$\theta = \sum_{i=1}^n \arctan \lambda_i.$$

We extend the following generalization of theorems of Yuan [Yua02][Yua06].

Theorem 2. *Let g be the metric induced on $\Gamma = (x, Du(x))$. Suppose that u is an entire solution of the fourth order Hamiltonian stationary equation*

$$(0.1) \quad \Delta_g \theta = 0$$

with

$$(0.2) \quad |\theta(u)| > (n-2) \frac{\pi}{2} + \delta$$

or

$$D^2u \geq 0.$$

Then u is a quadratic function, that is, Γ is a plane.

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For a fixed bounded domain $\Omega \subset \mathbb{R}^n$ consider the volume functional

$$F_\Omega(u) = \int_\Omega \sqrt{\det \left(I + (D^2u)^T D^2u \right)} dx.$$

A function u is critical for $F_\Omega(u)$ under compactly supported variations of the scalar function if and only if u satisfies the equation (0.1) c.f [SW03, Proposition 2.2]. In other words, the gradient graph of u has smallest volume compared with other gradient graphs. Recall that if u satisfies the special Lagrangian equation [HL82]

$$(0.3) \quad D\theta = 0$$

then u is critical under *all* variations of the surface.

The Liouville property, together with (0.2) will force θ to satisfy (0.3), so Γ is a minimal surface. It then follows immediatly from a result of Yuan that Γ is a plane. For Bernstein results for (0.1) with a volume growth constraint, and more discussion of the problem, see [Mes01].

1. PROOF

1.1. Proof of Theorem 1. If u is semiconvex, then there exists a value M such that

$$D^2u + MI_n \geq 0.$$

It follows that

$$\arctan \lambda_i \geq -\arctan M$$

for all λ_i .

Letting

$$\delta = \frac{\pi}{2} - \arctan M > 0,$$

and

$$D^2u \geq \tan \left(\delta - \frac{\pi}{2} \right) I.$$

The Yuan rotation from [Yua06, section 2] is as follows.

Consider the map

$$T(x) = \cos \left(\frac{\delta}{n} \right) x + \sin \left(\frac{\delta}{n} \right) Du(x).$$

Differentiating

$$\begin{aligned} (1.1) \quad DT &= \cos \left(\frac{\delta}{n} \right) I + \sin \left(\frac{\delta}{n} \right) D^2u(x) \\ &\geq \cos \left(\frac{\delta}{n} \right) I + \sin \left(\frac{\delta}{n} \right) \tan \left(\delta - \frac{\pi}{2} \right) I \\ &= \cos \left(\frac{\delta}{n} \right) \left(1 + \tan \left(\frac{\delta}{n} \right) \tan \left(\delta - \frac{\pi}{2} \right) \right) I. \end{aligned}$$

Recalling the formula

$$\tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}$$

we see

$$1 + \tan\left(\frac{\delta}{n}\right) \tan\left(\delta - \frac{\pi}{2}\right) = \frac{\tan\left(\frac{\delta}{n}\right) - \tan\left(\delta - \frac{\pi}{2}\right)}{\tan\left(\frac{\delta}{n} - \left(\delta - \frac{\pi}{2}\right)\right)} = \frac{\tan\left(\frac{\delta}{n}\right) + \tan\left(\frac{\pi}{2} - \delta\right)}{\tan\left(\frac{\pi}{2} - \delta\frac{n-1}{n}\right)}.$$

It follows that

$$DT \geq \cos\left(\frac{\delta}{n}\right) \frac{\tan\left(\frac{\delta}{n}\right) + \tan\left(\frac{\pi}{2} - \delta\right)}{\tan\left(\frac{\pi}{2} - \delta\frac{n-1}{n}\right)} I > 0,$$

and the map

$$T : \mathbb{R}^n \rightarrow \mathbb{R}^n$$

is a diffeomorphism.

Next consider the map

$$\tilde{D} = Du \circ T^{-1}.$$

By (1.1),

$$D\tilde{D}(y) = D^2u(T^{-1}(y)) \left[\cos\left(\frac{\delta}{n}\right) I + \sin\left(\frac{\delta}{n}\right) D^2u(T^{-1}(y)) \right]^{-1}.$$

Diagonalizing D^2u at $T^{-1}(y)$ we see

$$D\tilde{D}|_y \leq \max_i \frac{\lambda_i}{\cos\left(\frac{\delta}{n}\right) + \sin\left(\frac{\delta}{n}\right) \lambda_i} \leq \frac{1}{\sin\left(\frac{\delta}{n}\right)} = M_0 < \infty.$$

So the map

$$G : \mathbb{R}^n \rightarrow \Gamma \subset \mathbb{R}^n \times \mathbb{R}^n$$

given by

$$G(x) = (T^{-1}(x), Du \circ T^{-1})$$

is a diffeomorphism onto the gradient graph Γ . Thus the pulled-back metric is given by

$$g = I_n + G^* \bar{g}$$

and satisfies

$$I_n \leq g \leq (1 + M_0^2) I_n.$$

It follows that the Laplacian, given by

$$\Delta_g f = \frac{\partial_j (\sqrt{\det g} g^{ij} \partial_i f)}{\sqrt{\det g}},$$

is a uniformly elliptic divergence type operator.

The remaining proof is standard, but we include it for completeness.

We recall the Harnack inequality of De-Giorgi-Nash-Moser (cf [GT01, Theorem 8.20])

Theorem 3. *Let $u \geq 0$ be a solution of*

$$\partial_j (a^{ij}(x) \partial_i f(x)) = 0$$

on $B_4(0)$ with

$$0 < \varepsilon I_n \leq a^{ij} \leq \frac{1}{\varepsilon} I_n.$$

There exists a constant C such that

$$\sup_{B_1(0)} f \leq C \inf_{B_1(0)} f.$$

We may assume that either f or $(-f)$ is bounded below, and we may add a constant and assume $\inf f = 0$. Notice that

$$f_R(x) = f\left(\frac{x}{R}\right)$$

is a solution of the equation

$$\partial_j \left(a^{ij} \left(\frac{x}{R} \right) \partial_i f_R(x) \right) = 0$$

so satisfies the hypothesis of the Harnack inequality. In particular

$$\sup_{B_R(0)} f \leq C \inf_{B_R(0)} f$$

for every ball $B_R(0)$ with a fixed constant C . Taking $R \rightarrow \infty$ gives $\sup f = 0$.

In fact, we can state a slightly more general theorem.

Theorem 4. *Suppose that $F(D^2u)$ is an elliptic functional, and let g be the induced metric on the gradient graph. If u is semi-convex and*

$$\Delta_g F(D^2u) = 0$$

then

$$F(D^2u) = \text{const.}$$

Proof. If u is semiconvex, then there exists a value M such that

$$D^2u - MI_n \geq 0.$$

It follows by ellipticity that

$$F(D^2u) \geq F(MI_n) > -\infty.$$

The result follows immediately from our main theorem. □

1.2. Proof of Theorem 2. The function θ is odd in u , so we need only show that

$$(1.2) \quad \theta(u) > (n-2) \frac{\pi}{2} + \delta$$

implies that u is semiconvex. If

$$\lambda_i < \arctan(\delta - \frac{\pi}{2})$$

then we must have

$$\sum_{i \neq j} \arctan \lambda_j > (n-2) \frac{\pi}{2} + \delta - (\delta - \frac{\pi}{2}) = (n-1) \frac{\pi}{2}$$

which is clearly a contradiction as

$$\arctan \lambda_j \leq \frac{\pi}{2}.$$

We conclude that

$$D^2 u \geq \arctan(\delta - \frac{\pi}{2})$$

and u is semiconvex. Thus $\theta(u) = \text{const}$, by Theorem 1. The result follows by the main results in [Yua02][Yua06].

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